We describe the main features of an in-service teacher training proposal aimed at helping the teachers to use logic as a means of encouraging students to develop valuable models of thinking, taking into account the evolution of common sense reasoning and the reasoning capabilities required by the complexity of global rationality. Referring to the situation in Italian schools, we firstly discuss the levels of competence to be achieved by teachers, in relation to the formal reasoning objectives of education. Then, we introduce our proposal, giving particular emphasis to its epistemological design. We conclude with some remarks about the results of our experimentation with the model.

We propose innovations in teacher training in logic, in accordance with the current view of rationality. Our work is partly supported by the project ‘Education to rationality’ of IRRE Liguria (Ligurian Regional Institute of Educational Research), a project aimed at improving reasoning education in upper secondary schools.

Introduction

Mathematical logic constitutes the basis for rationally modelling common sense reasoning, thus offering a powerful tool to help students develop valuable models of thinking. To fully exploit its pedagogical value, it should be compared to common sense reasoning. Moreover, the teaching of mathematical logic should be connected, at least to some extent, to the complexity of global rationality, in which common sense clashes with the results of formal logic.

However, at least in the Italian school, the teaching of mathematical logic has not traditionally followed this approach: in the lower school, procedures, rules, and behaviours are taught, with little attention to reasoning; in the upper school, limited attention is paid to logic, as it usually is not included in the final examination. Only recently, the increasing need of renewing education on reasoning has led the Italian school to focus attention on the design of curricula aimed at helping students to reach awareness of logical tools which are part of the present culture and technology, and to acquire the expressive and linguistic capabilities related to them.
To create effective changes in schools, however, in-service teachers have to construct knowledge about the variety of formal logics at their disposal and to gain awareness of their possible influence on common sense reasoning.

In this chapter, we will discuss a plan for re-designing in-service teacher training in logical modelling, including a contemporary view of rationality. In fact, the limited attention to logic education is, in our opinion, a main cause of the decreasing capability of the Italian upper secondary school to provide a valuable education in reasoning. This situation is confirmed by a recent analysis carried out by IRRE Liguria via a questionnaire, addressed to teachers of all Ligurian upper secondary schools [Gentilini & Manildo, 2001].

The work is partly based on the project Education to Rationality of IRRE Liguria (Ligurian Regional Institute of Educational Research), a project aimed to improve education to reasoning in upper secondary schools [Gentilini, 2001].

Our teacher training proposal

Framework. Active involvement in current socio-cultural and technological processes requires that people be able to reason by choosing among different rationalities that apparently contradict each other, the appropriate one for the situation at hand. Global rationality, in fact, takes into account all of the following:

- **Common sense reasoning**: This includes the reasoning capabilities that help to select behaviours or arguments considered ‘obviously reasonable’ by the majority of people (including students), thus determining ‘obvious deductions’. It is difficult to define ‘obviously reasonable’ [Fagin Halpern, Moses & Vardi, 1995], as it depends on the time and on the cultural context.

- **Scientific reasoning**: This is the type of reasoning canonically established by the scientific world. It can be divided into two sub-categories: classical and modern rationality. Classical reasoning is based on the theory of the inference and on the theory of truth established by classical thought, essentially Aristotelian, without the use of any artificial languages. Modern rationality, dating from the end of the nineteenth century to the present, is dominated by mathematical logic. It is built upon symbolic-formal languages, on the mathematical approach to classical logic, and on non-standard logics.

Examples of conflicts between different kinds of rationalities are shown in Figure 1.

**Figure 1.** Some examples of conflicts between common sense and scientific reasoning

**Example 1**
Both the Euclidean straight line and plane include an infinite number of points. Are there more points on the plane or on the line? Common sense reasoning would answer that the plane includes more points than the line or that nothing can be said.

Cantor’s set theory proves that the two sets of point can be put in a bijective correspondence. Thus, the plane and the line are sets with the same infinite cardinality.
Example 2
Common sense reasoning would say: *High* and *Low* exist as absolute orientations in space.

Physics states: *Absolute High* and *Absolute Low* do not exist.

We note that, in everyday life, the common sense notions of *High* and *Low* are employed.

Example 3
As shown by experimental psychology [Evans, Newstead, & Byrne, 1993], common sense reasoning often leads people to make inferences of the following types:

**Premises**
If there is a circle on the left, then there is a square on the right.
There is not a circle on the left.

**Conclusion:** There is not a square on the right

**Premises**
If there is a circle on the left, then there is a square on the right.
There is a square on the right

**Conclusion:** There is a circle on the left

In both cases, the inferences applied are non valid in Classical Logic.

Thus, the goals of teaching reasoning should be to create awareness of the various types of rationality, and to develop strategies for handling the conflicts between them. To reach these objectives, new models for in-service teacher training on logic should be devised which are suitable for helping teachers to: 1) acquire knowledge of the logic underlying the different types of rationality that are part of our present culture; 2) use this knowledge to help students develop awareness of the types of rationality that are functioning in our present culture. These are the goals of the project *Education to Rationality* of IRRE Liguria. The project trains in-service upper secondary school teachers (mathematics, science, humanities and philosophy teachers) on mathematical logic.

**Overview.** Our proposal is organised in two parts, as follows.

- The first phase, oriented directly to the classroom work, is centred on the formal notion of logical consequence, and on possible instructional approaches to this notion.

- The second phase introduces teachers to formal rationality produced in the last century, and it addresses the modelling power of the complex global rationality of mathematical logic via a comparison between classical, intuitionistic and paraconsistent logic.
Each phase includes two kinds of activities: theoretical lessons and examples of rational arguments. Discussions are designed to help the teachers turn theory into meaningful classroom activity for learning.

Let us briefly analyse these phases.

**Theoretical lesson**

**Phase 1.** It includes 1) an introduction to modern classical logic tools including logical consequence and 2) the analysis of heuristic methods suitable for teaching this notion.

1. We propose the following topics, that could be partly adapted also for students of the courses of mathematics and philosophy (last year of upper secondary school):

   - Introduction to the idea of artificial language, via the formal-symbolic logic language of the Predicate Logic $\text{LK}$. A rigorous approach, appropriate for teachers of different backgrounds, can be inspired, for example, by Barwise and Etchemendy, (1992).

   - A presentation of the deductive apparatus of $\text{LK}$, possibly via Gentzen’s natural deduction and an examination of the distinction between syntax and semantics for a theory $T$ axiomatized in the language of $\text{LK}$ and basic definitions of the Tarski semantics for the language of $\text{LK}$.

   - Presentation of the notion of logical consequence.

   - Analysis of the links between syntax and semantics for the first order logic $\text{LK}$, via the result of completeness for $\text{LK}$.

2. In teaching heuristics, we observe that a number of pedagogical difficulties have to be overcome to introduce to the students the notion of logical consequence, in particular the concept of semantics of a symbolic language. A (rigorous) heuristic approach, and its adaptation to arguments expressed by using the natural language, can help to overcome the problem. Thus, teachers can introduce the idea of logical consequence even if students do not know the formal language of $\text{LK}$.

   Accordingly, in our training proposal we introduce the concept of possible world, corresponding to a Tarski formal interpretation, that is:

   A possible world $\Omega$ interpreting a statement $B$ is a state of affairs alternative to the real world of references such that: i) $\Omega$ is endowed with an intrinsic factual coherence; ii) and it includes objects and situations to which terms and predicates occurring in $B$ can be referred.

   For classroom practice, the main difficulty is the introduction of the following definition: Proposition $C$ is a logical consequence of hypothesis $B_1,...,B_n$ if in every possible world (or circumstance) where $B_1,...,B_n$ hold, also $C$ holds. This definition has to be discussed also by using examples deriving from arguments expressed in natural language.

**Phase 2.** This phase helps teachers to experiment with different types of sound deduction and truth notions, and with situations showing the use of different logics in the same context at different levels. In the classroom, teachers can apply this experience mainly by presenting meaningful examples to the students.

The content of this phase includes:

1. A review of classical logic in comparison to other logics. In particular classical logic can be presented as a logic oriented towards ontology and might be used to describe mathematical entities within a Platonist conception of mathematics. This presentation, moreover, helps to point out that the lack of awareness of classical deduction rules and diffused tautologies (for example the excluded middle principle, the double negation law, the non contradiction principle) prevents understanding of non-classical logics.
2. Intuitionistic logic [Troelstra & Van Dalen, 1988]. This topic is introduced to show teachers a formal view of logic different from the classical one, and to help them to experiment with the possibility of applying different logics to the same problem. More specifically, the content includes:

- Presentation of historical and philosophical roots of intuitionism, a logic centred on constructive proofs as objects produced by the human mind.
- A discussion of the lack of validity of the excluded middle principle and of the double negation law.
- A presentation of the Heyting’s axiomatization of the formal system for the intuitionistic predicate calculus $LJ$, limited to pure predicate logic, without function symbols.
- An essential presentation of Kripke models for $LJ$, with particular emphasis on the examples of falsification of classical tautologies [(Van Dalen, 1986)]. This training step is particularly important as it proposes a different notion of the truth, with respect to that of Tarski for the classical logic.
- An example of a mathematical reasoning which employs intuitionistic logic at the theory level and classical logic at the meta-theory level. The example is reported in Appendix 1.

3. An introduction to Paraconsistent Logic [Da Costa, 1974]. and the role that logic can play in modelling the construction of relevant conjectures in scientific discovery.

In fact, paraconsistent predicate logic systems can be extended by means of contradictions $F \land \neg F$ without trivializing. Thus, in the semantics of paraconsistent logic, a contradiction is not necessarily false. This characteristic makes paraconsistent rationality a powerful tool to model conjectures.

In fact, we observe that the most interesting and innovative conjectures with respect to a well-established theory are often contradictions from the point of view of the theory. Let us consider for example Classical Mechanics and General Relativity. They are classically mutually inconsistent. If we include in classical mechanics Galileo’s principle of addition of velocities, we can derive from classical mechanics that “a light signal exists having a velocity greater than $c$” while from general relativity we obviously derive that “light signals having speed greater than $c$ do not exist”. Therefore, the simultaneous application of the two theories produces a contradiction. However, we cannot simply say that the relativity postulates trivialize classical mechanics.

The program of the part about paraconsistency includes:

- An introduction of a paraconsistent logical system, for example the system C1 of Da Costa (1974).
- An example of application via the modelling of the ‘break’ of Lagrangian Mechanics with respect to Newtonian Mechanics. The example is reported in Appendix 2.

**Discussions**

**Phase 1.** We report here some examples of the exercises that might be used to help teachers understand the theory.

a) Translation of sentences of the natural language into the language of predicate logic, and analysis of the representational power of artificial languages.

**Example of exercise**
Translate the statements:

*If a teacher teaches Logic to students then he has to be brave; Each minute a person is robbed in Genoa.*

Then, show that other sentences expressed in natural language, and presenting similar structures, can be represented by means of the translations obtained.

b) Idea of possible world.

**Example of exercise**

Discuss the fact that the statement:

*Napoleon Bonaparte died in 1821*

admits possible worlds where it is false, according to the heuristic definition of possible world.

c) Notion of logical consequence.

**Example of exercise**

By using the heuristic notion of possible world, discuss the fact that, in the following arguments expressed in natural language, the conclusion is a logical consequence of the premises.

1. *All mammals have lungs. All whales are mammal. Thus, all whales have lungs.*
2. *All individual with 10 paws have wings. All spiders have 10 paws. Thus, all spiders have wings.*

Discuss then the following facts: in argument 1) all propositions are true with respect to the ‘standard’ world; in argument 2) all propositions are false with respect to the ‘standard’ world.

Why these facts do not influence the existing relation of *logical consequence*?

d) Deductive apparatus of **LK**

**Example of exercise**

Given the following syllogism, expressed in natural language:

*Every budgie is an animal with wings*
*All animals with wings are white*
*Every budgie is white*

Verify that a translation into the language of the Predicate logic is:

\[
\forall x(C(x) \rightarrow \Lambda(x)) \\
\forall y(\Lambda(y) \rightarrow B(y)) \\
\forall z(C(z) \rightarrow B(z))
\]

On the basis of the inference rules of the Natural Deduction for **LK**, show that:

1) a proof of the conclusion from the hypotheses exists; 2) the conclusion, in the context of the classical logic, is a logical consequence of the hypotheses.

e) Notion of logical consequence.
Example of exercise
Show that the following argument, expressed in natural language:

*If I had all the gold of Fort Knox I would be rich. I do not have all the gold of Fort Knox. Thus, I am not rich.*

does not establish a relation of logical consequence between the premises and the conclusion.

Phase 2. Discussions mainly focus on the criticism of arguments expressed in natural language, from the points of view of modern logic. Various perspectives are taken into account: classic, intuitionistic and paraconsistent. Let us show an example carried out with the teachers.

Example
Let us consider the following argument expressed in natural language:

*If individuals are good, laws are not necessary to prevent them from acting badly; on the contrary, if individuals are bad, laws will not be able to avoid the fact that they will act badly. Thus the laws are not necessary, or useless.*

An analysis of the argument was so developed with the teachers.

The argument is implicitly based on the equivalence between *not-good* and *bad*. A partial syntactic reduction transforms the initial argument into:

Hypotheses:  
*individuals are good → laws are not necessary; individuals are not-good → laws are useless*  

Logical axiom:  
*(individuals are good) or (individuals are not-good)*  
*(laws are not necessary) or (laws are useless)*

**Syntactic analysis.** The argument is correct from the point of view of classical logic. In fact, from the hypotheses we deduce:

* (individuals are good) or (individuals are not-good) → (laws are not necessary) or (laws are useless).

Then, by applying the logical rule of *modus ponens* with the axiom of the *excluded middle* we have the thesis.

**Semantic analysis.** However, we can criticise the argument from the semantic point of view. In fact, it can be argued that the hypothesis *individuals are not-good → laws are useless* is false in all possible worlds, or that there are not possible worlds in which both hypotheses are simultaneously true. Thus, the hypotheses constitute a contradiction. Therefore, the argument is correct but useless, as we can prove everything from contradictory hypotheses.

**Comparison with other logics.** It can be observed that, according to the intuitionistic view, the syntactic analysis illustrated above does not work, as the axiom of the excluded-middle cannot be accepted. Moreover, the argument refers to the potentially infinite set of all possible individuals.

From the paraconsistent point of view, finally, the semantic criticism above carried out is not possible, since in Paraconsistent Logic, in general, from a contradiction it is not possible to derive every sentence.
Comments on the teaching experiment. The course outlined above was used in a training experiment, in the scholastic year 2000-2001, organised by IRRE Liguria. Twenty upper secondary school in-service teachers (mathematics, science, philosophy, humanities teachers) took part in the course. The course included 24 hours of lessons and about 20 hours of discussions of arguments, aimed at producing instructional modules. The teachers were given a questionnaire to determine their opinions about both the reasoning capabilities of their students and the changes they thought they needed to make in their teaching to improve student reasoning. Thus, teachers were motivated to take part in the course, as a way to address perceived problems. The reaction of the teachers was quite positive. They showed interest in the theoretical notions introduced, notwithstanding the difficulties they faced. There were two main difficulties:

1. It was difficult for the participants to understand that the syntactic and the semantic analysis of a reasoning process are independent of each other. We observed that the majority of teachers who did not have prerequisites modern logic, often confused the truth and the syntactic correctness. A typical misunderstanding of this kind, we observed, was the belief that a deduction is not sound if all sentences in the deduction are false with respect to the ‘standard’ world.

2. The teachers also had difficulty accepting a variety of types of deductions and truths. Teachers showed perplexity about the existence of various logics, all valuable, and about the possibility of choosing the one to adopt according to the situation at hand.

The examples we discussed were valuable in helping teachers to overcome these difficulties, and, at the same time, they formed a basis for devising instructional activities for students. In fact, as a result of the course, teachers developed some modules that they intended to use with their students. An example is given in the following section.

Relating the proposal to classroom practice

One module was designed to help students in the first two years of upper secondary school (14-16 years old), develop competence with proofs. Typically, there is little development of reasoning abilities at his age.

The module used an interdisciplinary approach to basic linguistics and mathematics. Students are encouraged to reflect on the knowledge underlying the procedures introduced, by 1) giving written (linguistic) explanations of their meaning; 2) acquiring awareness of the variety of uses of some terms; 3) inventing and writing elementary mathematics proofs. Explanations are based upon definitions, theorems (in scientific arguments natural language is used), and axioms. Terms sometimes have a twofold meaning: they can be viewed as notions of a specific discipline or as words of the natural language (e.g., energy, straight line, product). Thus, the students’ mathematical proofs are to be analysed by the humanities teachers for syntactic correctness as well as the construction of a valid argument.

Examples of the activities to develop with students are shown in Figure 2. The work is currently being used in two Ligurian schools.
Activity 1
What does ‘To compare two numbers’ mean? Write two fractions, of your choice. Consider various methods to compare them. Write an explanation of these methods and apply them to the fractions at hand.

Activity 2
The word set refers to a group of objects which exhibit a common property. In mathematics, a set is defined if this property can be defined avoiding ambiguity and contradiction. Taking into account this fact, indicate which expressions, among the following, define a set in mathematical sense, and explain why. Expressions: 1) the most important Italian towns; 2) the Italian towns which are provincial capitals; 3) the ‘small’ numbers; 4) the numbers that can be divided by 3; 5) the active students of a classroom.

Activity 3
Consider the definition: If an integer number $m$ is the product of an integer number $n$ and another integer number (different from 0) then we say that $m$ is multiple of $n$.
Write a proof of the following propositions: 1) 45 is multiple of 15; 2) 45 is not multiple of 6.

References
Appendix 1. Using classical logic to prove theorems about intuitionistic logic

Prove that \( \neg \forall x (Q(x) \lor \neg Q(x)) \) (a double negation of an instance of the excluded middle principle) can be falsified in Kripke semantics for intuitionistic logic. (for the presentation of Kripke models see [Van Dalen 1986]).

Proof. Let us consider the model K: \( \{ w_0, w_1, w_2, w_3, \ldots \} \), with an infinite number of nodes \( w_0 \leq w_1 \leq w_2 \leq \ldots \). For every \( i \geq 0 \), the domain \( D(w_i) \) is the set \( \{ 0, \ldots , i \} \). The formula \( Q(i) \) can be asserted in every \( w_j, j > i \); at the node \( w_0 \) no atomic statement that can be asserted.

We have the following heuristic scheme:

\[
\begin{align*}
    w_3 & \{ 0,1,2,3 \} \quad Q(0), Q(1), Q(2) \\
    w_2 & \{ 0,1,2 \} \quad Q(0), Q(1) \\
    w_1 & \{ 0,1 \} \quad Q(0) \\
    w_0 & \{ 0 \}
\end{align*}
\]

Let us suppose: \( w_p \models \neg \forall x (Q(x) \lor \neg Q(x)) \)

Then, \( \neg \forall x (Q(x) \lor \neg Q(x)) \) cannot be asserted in any node of the model. Thus, for every node \( w_p \), a node \( w_t \), \( w_t \geq w_p \), exists, such that \( \forall x (Q(x) \lor \neg Q(x)) \) can be asserted in \( w_t \). Let us suppose that \( \forall x (Q(x) \lor \neg Q(x)) \) can be asserted in \( w_t \).

In particular, we have: \( w_t \models Q(t) \lor \neg Q(t) \).

By construction \( Q(t) \) cannot be asserted in \( w_t \).

Moreover, \( w_t \models \neg Q(t) \) does not hold, as, by construction: \( w_t \models Q(t) \).

Thus, it does not exist \( w_t \geq w_p \) such that \( \forall x (Q(x) \lor \neg Q(x)) \) can be asserted in \( w_t \).

Thus, \( \neg \forall x (Q(x) \lor \neg Q(x)) \) cannot be asserted in \( w_0 \).

From a cognitive point of view, the main difficulty here concerns the possibility of using classical logic (reduction to absurdity and the principle of the excluded middle) to prove results about intuitionistic logic, without provoking lack of coherence in the overall reasoning.

D’Alembert principle includes a contradiction, but classical mechanics does not trivialize

Let us consider a physical system of particles $m_i$, $i=1,2,\ldots,n$. The virtual works principle states that:

$$\sum_i F_i \delta r_i = 0$$

where: $\delta r_i$: virtual shift of the particle $m_i$ that is a shift “without time”; $F_i$: the active forces.

That is: in the equilibrium state the virtual work of the active forces must be zero.

In a dynamic state, (with impulses $p_i = m_i v_i$ ) the principle above becomes the D’Alembert principle:

$$\sum_i (F_i- \dot{p}_i) \delta r_i=0$$

where: $F_i$: real forces; $\dot{p}_i$: opposite forces; (by the notion of virtual shift dynamics is reduced to statics).

The D’Alembert principle allows us to define the Lagrange equations of the system:

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{q}_i} \right) - \frac{\delta L}{\delta q_i} = 0$$

where: $L = T-V$, $T$: kinetic energy; $V$: potential energy (see [Goldstein 1965]).

We remark that the virtual works principle is a very productive principle but it is a strong conjecture w.r.t. Newton Mechanics since the central notion of virtual shift implies a contradiction.

In fact, by assuming that in Newton Mechanics only one absolute time $t$ is admitted, we have:

$\delta r_i$ virtual requires: $\frac{d}{dt} (\delta r_i ) =0$

however, $\delta r_i$ is a shift requires: $\frac{d}{dt} (\delta r_i ) >0$

thus, the principle must be formulated:

$$(\sum_i F_i \delta r_i = 0 ) \land ( \frac{d}{dt} (\delta r_i ) =0 ) \land ( \frac{d}{dt} (\delta r_i ) >0 )$$

which implies a contradiction, since $\frac{d}{dt} (\delta r_i ) >0$ implies $\frac{d}{dt} (\delta r_i ) = 0$.

Note. It could be observed that the common-sense use of the results of theoretical classical mechanics must have some paraconsistent reasoning resources.